

# On the Sample Complexity of Learning Sparse Graphical Games

Jean Honorio  
 Computer Science, Purdue University  
 West Lafayette, IN 47907, USA  
 jhonorio@purdue.edu

## Abstract

We analyze the sample complexity of learning sparse graphical games from purely behavioral data. That is, we assume that we can only observe the players' joint actions and not their payoffs. We analyze the sufficient and necessary number of samples for the correct recovery of the set of pure-strategy Nash equilibria (PSNE) of the true game. Our analysis focuses on sparse directed graphs with  $n$  nodes and at most  $k$  parents per node. By using VC dimension arguments, we show that if the number of samples is greater than  $\mathcal{O}(kn \log^2 n)$ , then maximum likelihood estimation correctly recovers the PSNE with high probability. By using information-theoretic arguments, we show that if the number of samples is less than  $\Omega(kn \log^2 n)$ , then any conceivable method fails to recover the PSNE with arbitrary probability.

## 1 Introduction

Non-cooperative game theory has been considered as the appropriate mathematical framework in which to formally study *strategic* behavior in multi-agent scenarios. The core solution concept of *Nash equilibrium* serves a descriptive role of the stable outcome of the overall behavior of self-interested agents (e.g., people, companies, governments, groups or autonomous systems) interacting strategically with each other in distributed settings.

**Applications.** There has been considerable progress on computing Nash equilibria and related problems in the context of graphical games, as well as several applications. In *political science* for instance, the work of [5] identified the most influential senators in the U.S. congress (i.e., a small set of senators whose collectively behavior forces every other senator to a unique choice of vote). The most influential senators were intriguingly similar to the gang-of-six senators formed during the national debt ceiling negotiation in 2011. Additionally, it was observed in [4] that the influence from Obama to Republicans increased in the last sessions before candidacy, while McCain's influence to Republicans decreased.

**Learning Sparse Games.** The *strategic* inference problems above (i.e., computing Nash equilibria or finding the most influential agents) require a known graphical game which is unobserved in the real world. To overcome this issue, learning graphical games from behavioral data was proposed in [4], by using maximum likelihood estimation (MLE) and *sparsity*-promoting methods. We also note that [4, 5] have shown the successful use of *sparse* graphical games in real-world settings, such as the analysis of the U.S. congressional voting records as well as the U.S. supreme court.

**Contributions.** In this paper, we study the *statistical* aspects of the problem of learning graphical games from strictly behavioral data. As in [4], we assume that we can only observe

the players’ joint actions and not their payoffs. The class of models considered here are linear influence games, previously introduced in [5]. We study the sufficient and necessary number of samples for the correct recovery of the pure-strategy Nash equilibria (PSNE) set of the true game, for sparse directed graphs with  $n$  nodes and at most  $k$  parents per node. Theorem 3 shows that the sufficient number of samples for MLE is  $\mathcal{O}(kn \log^2 n)$ . Theorem 4 shows that the necessary number of samples for any conceivable method is  $\Omega(kn \log^2 n)$ . Thus, MLE is statistically optimal.

**Discussion.** While *sparsity*-promoting methods were used in prior work [4], the benefit of sparsity for learning games has not been theoretically analyzed before. In this paper, we focus on the *statistical* analysis of exact MLE.<sup>1</sup> Prior work has not focused on the correct PSNE recovery, but on generalization bounds. More formally, Corollary 15 in [4] shows that for general (possibly dense) graphs with  $n$  nodes,  $\mathcal{O}(n^3)$  samples are sufficient for the empirical MLE minimizer to be close to the best achievable expected log-likelihood. As a byproduct of our PSNE recovery analysis, we also provide better generalization bounds. More specifically, Lemma 2 shows that for  $k$ -sparse graphs, only  $\mathcal{O}(kn \log^2 n)$  samples are sufficient for obtaining a good expected log-likelihood.

## 2 Graphical Games

In classical game-theory (see, e.g. [3] for a textbook introduction), a *normal-form game* is defined by a set of *players*  $V$  (e.g., we can let  $V = \{1, \dots, n\}$  if there are  $n$  players), and for each player  $i$ , a set of *actions*, or *pure-strategies*  $A_i$ , and a payoff function  $u_i : \times_{j \in V} A_j \rightarrow \mathbb{R}$  mapping the joint actions of all the players, given by the Cartesian product  $\mathcal{A} \equiv \times_{j \in V} A_j$ , to a real number. In non-cooperative game theory we assume players are greedy, rational and act independently, by which we mean that each player  $i$  always want to maximize their own utility, subject to the actions selected by others, irrespective of how the optimal action chosen help or hurt others.

A core solution concept in non-cooperative game theory is that of an *Nash equilibrium*. A joint action  $\mathbf{x}^* \in \mathcal{A}$  is a *pure-strategy Nash equilibrium (PSNE)* of a non-cooperative game if, for each player  $i$ ,  $x_i^* \in \arg \max_{x_i \in A_i} u_i(x_i, \mathbf{x}_{-i}^*)$ ; that is,  $\mathbf{x}^*$  constitutes a *mutual best-response*, no player  $i$  has any incentive to unilaterally deviate from the prescribed action  $x_i^*$ , given the joint action of the other players  $\mathbf{x}_{-i}^* \in \times_{j \in V - \{i\}} A_j$  in the equilibrium. In what follows, we denote a game by  $\mathcal{G}$ , and the set of all *pure-strategy Nash equilibria* of  $\mathcal{G}$  by

$$\mathcal{NE}(\mathcal{G}) \equiv \{\mathbf{x}^* \mid (\forall i \in V) x_i^* \in \arg \max_{x_i \in A_i} u_i(x_i, \mathbf{x}_{-i}^*)\}.$$

A (*directed*) *graphical game* is a game-theoretic graphical model [6]. It provides a succinct representation of normal-form games. In a graphical game, we have a (directed) graph  $G = (V, E)$  in which each node in  $V$  corresponds to a player in the game. The interpretation of the edges/arcs  $E$  of  $G$  is that the payoff function of player  $i$  is only a function of the set of parents/neighbors  $\mathcal{N}_i \equiv \{j \mid (i, j) \in E\}$  in  $G$  (i.e., the set of players corresponding to nodes that point to the node corresponding to player  $i$  in the graph). In the context of a graphical game, we refer to the  $u_i$ ’s as the *local payoff functions/matrices*.

*Linear influence games (LIGs)* [4, 5] are a sub-class of 2-action graphical games with *parametric* payoff functions. For LIGs, we assume that we are given a matrix of influence weights  $\mathbf{W} \in \mathbb{R}^{n \times n}$ , with  $\text{diag}(\mathbf{W}) = \mathbf{0}$ , and a threshold vector  $\mathbf{b} \in \mathbb{R}^n$ . For each player  $i$ , we define the payoff function  $u_i(\mathbf{x}) \equiv x_i(\sum_{j \in \mathcal{N}_i} w_{ij}x_j - b_i) = x_i(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i)$ . We further assume binary actions:  $A_i \equiv \{-1, +1\}$  for all  $i$ . The *best response*  $x_i^*$  of player  $i$  to the joint action  $\mathbf{x}_{-i}$  of the other players is defined as

---

<sup>1</sup> We leave the analysis of computationally efficient methods for future work. To put this in context, note that theoretical analysis for learning Bayesian networks has focused exclusively on exact MLE [1].

$$\begin{aligned}
\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} > b_i &\Rightarrow x_i^* = +1, \\
\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} < b_i &\Rightarrow x_i^* = -1, \\
\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} = b_i &\Rightarrow x_i^* \in \{-1, +1\},
\end{aligned}$$

or equivalently

$$x_i^* (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) \geq 0. \quad (1)$$

*Intuitively*, for any other player  $j$ , we can think of  $w_{ij} \in \mathbb{R}$  as a *weight* parameter quantifying the “influence factor” that  $j$  has on  $i$ , and we can think of  $b_i \in \mathbb{R}$  as a *threshold* parameter quantifying the level of “tolerance” that player  $i$  has for playing  $-1$ .

As noted in [4], games with different graph structure and weights can induce the same PSNE sets. For instance, let

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{W}' = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{W}'' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Define the LIGs  $\mathcal{G} = (\mathbf{W}, \mathbf{0})$ ,  $\mathcal{G}' = (\mathbf{W}', \mathbf{0})$  and  $\mathcal{G}'' = (\mathbf{W}'', \mathbf{0})$ . These three games defined above induce the same PSNE sets, i.e.,  $\mathcal{NE}(\mathcal{G}) = \mathcal{NE}(\mathcal{G}') = \mathcal{NE}(\mathcal{G}'') = \{(-1, -1, -1), (+1, +1, +1)\}$ .

**Equivalence Classes.** Each PSNE set defines an *equivalence class* of LIGs for which players have the same joint behavior. Thus, as argued further in Section 4 in [4], it is not possible to recover the structure and weights of the true game from observed joint actions. Instead, we can recover the PSNE set (or equivalence class) of the true game. Here, we study the sufficient and necessary number of samples for the correct recovery of the PSNE set of the true game.

### 3 Learning Graphical Games

In this paper, we define  $\mathcal{H}$  to be the class of LIGs with  $n$  nodes and  $k$  parents per node, as follows<sup>2</sup>

$$\mathcal{H} \equiv \left\{ \mathcal{G} \mid \mathcal{G} = (\mathbf{W}, \mathbf{b}) \wedge \mathbf{W} \in \mathbb{R}^{n \times n} \wedge \mathbf{diag}(\mathbf{W}) = \mathbf{0} \wedge \mathbf{b} \in \mathbb{R}^n \wedge \left( \forall i, \sum_j 1[w_{ij} \neq 0] \leq k \wedge |\mathcal{NE}(\mathcal{G})| \in \{1, \dots, 2^n - 1\} \right) \right\}.$$

The following simple generative model for joint actions was proposed in [4]. Let  $\mathcal{G}$  be a game, and let  $\mathcal{Q}_{\mathcal{G}}$  be a set defined as follows<sup>3</sup>

$$\mathcal{Q}_{\mathcal{G}} \equiv \left( \frac{|\mathcal{NE}(\mathcal{G})|}{2^n}, 1 - \frac{1}{2^{n+1}} \right).$$

With some probability  $q \in \mathcal{Q}_{\mathcal{G}}$ , a joint action  $\mathbf{x}$  is chosen uniformly at random from  $\mathcal{NE}(\mathcal{G})$ ; otherwise,  $\mathbf{x}$  is chosen uniformly at random from its complement set  $\{-1, +1\}^n - \mathcal{NE}(\mathcal{G})$ . Hence, the generative model is a mixture model with mixture parameter  $q$  corresponding to the probability that a stable outcome (i.e., a PSNE) of the game is observed, uniform over PSNE. Formally, the probability mass function (PMF) over joint-behaviors  $\{-1, +1\}^n$  parameterized by  $(\mathcal{G}, q)$  is

$$p_{\mathcal{G},q}(\mathbf{x}) \equiv q \frac{1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})]}{|\mathcal{NE}(\mathcal{G})|} + (1 - q) \frac{1[\mathbf{x} \notin \mathcal{NE}(\mathcal{G})]}{2^n - |\mathcal{NE}(\mathcal{G})|}, \quad (2)$$

where we can think of  $q$  as the “signal” level, and thus  $1 - q$  as the “noise” level in the data. Additionally,  $\mathcal{P}_{\mathcal{G},q}$  denotes the probability distribution defined by the PMF  $p_{\mathcal{G},q}(\cdot)$ .

<sup>2</sup>  $|\mathcal{NE}(\mathcal{G})| \in \{1, \dots, 2^n - 1\}$  comes from Definition 4 in [4].

<sup>3</sup>  $q \in (\frac{|\mathcal{NE}(\mathcal{G})|}{2^n}, 1)$  comes from Proposition 5 and Definition 7 in [4].

By using the PMF in eq.(2), we can define a (scaled) negative log-likelihood function over joint-behaviors  $\{-1, +1\}^n$  for a game  $\mathcal{G}$  and mixture parameter  $q$  as follows

$$\begin{aligned}\mathcal{L}_{\mathcal{G},q}(\mathbf{x}) &\equiv -\frac{\log p_{\mathcal{G},q}(\mathbf{x})}{(2n+1)\log 2} \\ &= -\frac{1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})]}{(2n+1)\log 2} \log \frac{q}{|\mathcal{NE}(\mathcal{G})|} - \frac{1[\mathbf{x} \notin \mathcal{NE}(\mathcal{G})]}{(2n+1)\log 2} \log \frac{1-q}{2^n - |\mathcal{NE}(\mathcal{G})|} .\end{aligned}\quad (3)$$

Note that since we scale the negative log-likelihood with a factor  $1/((2n+1)\log 2)$  then  $\mathcal{L}_{\mathcal{G},q}(\mathbf{x}) \in [0, 1]$  for all  $\mathcal{G} \in \mathcal{H}$ ,  $q \in \mathcal{Q}_{\mathcal{G}}$  and  $\mathbf{x} \in \{-1, +1\}^n$ .

Maximum likelihood estimation (MLE) allows to infer the game (and mixture parameter) from observed joint actions. More formally, given a dataset  $S$  of  $m$  joint actions, the *empirical* MLE minimizer is

$$(\hat{\mathcal{G}}, \hat{q}) = \arg \min_{\mathcal{G} \in \mathcal{H}, q \in \mathcal{Q}_{\mathcal{G}}} \frac{1}{m} \sum_{\mathbf{x} \in S} \mathcal{L}_{\mathcal{G},q}(\mathbf{x}) .$$

Assume that a joint action  $\mathbf{x}$  is drawn from an arbitrary data distribution  $\mathcal{D}$ . The *expected* MLE minimizer is given by

$$(\bar{\mathcal{G}}, \bar{q}) = \arg \min_{\mathcal{G} \in \mathcal{H}, q \in \mathcal{Q}_{\mathcal{G}}} \mathbb{E}_{\mathcal{D}}[\mathcal{L}_{\mathcal{G},q}(\mathbf{x})] .$$

Note that if the data is generated by a *true* game  $\mathcal{G}^* \in \mathcal{H}$  and mixture parameter  $q^* \in \mathcal{Q}_{\mathcal{G}^*}$ , then the expected MLE minimizer is the true game and mixture parameter. That is, if  $\mathcal{D} = \mathcal{P}_{\mathcal{G}^*,q^*}$  then  $\mathcal{NE}(\bar{\mathcal{G}}) = \mathcal{NE}(\mathcal{G}^*)$  and  $\bar{q} = q^*$ .

## 4 Sufficient Samples for PSNE Recovery

In this section, we show that if the number of samples is greater than  $\mathcal{O}(kn \log^2 n)$ , then MLE correctly recovers the PSNE with high probability.

**Number of PSNE Sets.** First, we show that the number of PSNE sets induced by LIGs is  $\mathcal{O}(2^{kn \log^2 n})$ . These results will be useful later in obtaining a generalization bound as well as for analyzing the correct recovery of PSNE.

**Lemma 1** (Number of PSNE sets). *Let  $\mathcal{H}$  be the class of LIGs with  $n$  nodes and  $k \in \{2, \dots, \lfloor \frac{9}{20}\sqrt{n} \rfloor - 1\}$  parents per node. Let  $d(\mathcal{H})$  be the number of PSNE sets that can be produced by games in  $\mathcal{H}$ , i.e.,  $d(\mathcal{H}) = |\cup_{\mathcal{G} \in \mathcal{H}} \{\mathcal{NE}(\mathcal{G})\}|$ . We have that  $d(\mathcal{H}) \leq 2^{4(k+1)n(\log^2 n + \log n)}$ .*

*Proof.* For every joint action  $\mathbf{x}$  and player  $i$ , define  $\mathbf{y}_i \equiv (x_i \mathbf{x}_{-i}, -x_i) \in \{-1, +1\}^n$ . For every LIG  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$ , we can rewrite the best response in eq.(1) as follows

$$x_i(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) = (\mathbf{w}_{i,-i}, b_i)^T \mathbf{y}_i \geq 0 .$$

Define the function class  $\mathcal{H}_i$  as follows

$$\mathcal{H}_i \equiv \left\{ f : \{-1, +1\}^n \rightarrow \{0, 1\} \mid f(\mathbf{y}_i) = 1[(\mathbf{w}_{i,-i}, b_i)^T \mathbf{y}_i \geq 0] \wedge \begin{matrix} (\mathbf{w}_{i,-i}, b_i) \in \mathbb{R}^n \\ \sum_j 1[w_{ij} \neq 0] \leq k \end{matrix} \right\} .$$

Note that  $\mathcal{H}_i$  is the class of linear classifiers in  $n$  dimensions, of sparse weight vectors  $(\mathbf{w}_{i,-i}, b_i)$  with at most  $k+1$  nonzero elements. By Theorem 20 in [7], the *Vapnik-Chervonenkis (VC) dimension* of  $\mathcal{H}_i$  is bounded as follows

$$\mathbb{VC}(\mathcal{H}_i) \leq 2(k+1) \log n .$$

The above requires that  $k+1 \in \{3, \dots, \lfloor \frac{9}{20}\sqrt{n} \rfloor\}$  and it only assumes  $\mathbf{y}_i \in \mathbb{R}^n$  instead of  $\mathbf{y}_i \in \{-1, +1\}^n$ . Define the boolean function  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  as follows

$$g(z_1, \dots, z_n) \equiv \prod_i z_i .$$

Note that if  $f_i \in \mathcal{H}_i$  for all  $i$ , then

$$g(f_1(\mathbf{y}_1), \dots, f_n(\mathbf{y}_n)) = 1 \Leftrightarrow \mathbf{x} \in \mathcal{NE}(\mathcal{G}) .$$

Define the function class

$$g(\mathcal{H}_1, \dots, \mathcal{H}_n) \equiv \{g(f_1(\mathbf{y}_1), \dots, f_n(\mathbf{y}_n)) \mid (\forall i) f_i \in \mathcal{H}_i\} .$$

By Lemma 2 in [8] we have that

$$\begin{aligned} \mathbb{VC}(g(\mathcal{H}_1, \dots, \mathcal{H}_n)) &\leq 2n(1 + \log n) \max_i \mathbb{VC}(\mathcal{H}_i) \\ &\leq 4(k+1)n(\log^2 n + \log n) . \end{aligned}$$

Finally, note that our analysis of  $\mathbb{VC}(g(\mathcal{H}_1, \dots, \mathcal{H}_n))$  gives a bound with respect to PSNE, while we are interested on PSNE sets. Thus,  $d(\mathcal{H}) \leq 2^{\mathbb{VC}(g(\mathcal{H}_1, \dots, \mathcal{H}_n))}$  and we prove our claim.  $\square$

**Generalization Bound.** Next, we show that if the number of samples is greater than  $\mathcal{O}(kn \log^2 n)$ , then the empirical MLE minimizer is close to the best achievable expected log-likelihood.

**Lemma 2** (Generalization bound). *Fix  $\delta, \varepsilon \in (0, 1)$ . Let  $\mathcal{H}$  be the class of LIGs with  $n$  nodes and  $k \in \{2, \dots, \lfloor \frac{9}{20}\sqrt{n} \rfloor - 1\}$  parents per node. Assume an arbitrary data distribution  $\mathcal{D}$ . Assume that  $S$  is a dataset of  $m$  joint actions (of the  $n$  players), each independently drawn from  $\mathcal{D}$ . If  $m \geq \frac{2}{\varepsilon^2}((4(k+1)n(\log^2 n + \log n) + 2) \log 2 + \log \frac{1}{\delta})$  then*

$$\mathbb{P}_S[\mathbb{E}_{\mathcal{D}}[\mathcal{L}_{\hat{\mathcal{G}}, \hat{q}}(\mathbf{x}) - \mathcal{L}_{\bar{\mathcal{G}}, \bar{q}}(\mathbf{x})] \leq \varepsilon] \geq 1 - \delta .$$

*Proof.* For clarity, let  $\mathcal{L}_S(\mathcal{G}, q) \equiv \frac{1}{m} \sum_{\mathbf{x} \in S} \mathcal{L}_{\mathcal{G}, q}(\mathbf{x})$  and  $\mathcal{L}_{\mathcal{D}}(\mathcal{G}, q) \equiv \mathbb{E}_{\mathcal{D}}[\mathcal{L}_{\mathcal{G}, q}(\mathbf{x})]$ . By Lemma 11 in [4], for any game  $\mathcal{G}$  and for  $0 < q'' < q' < q < 1$ , if for any  $\varepsilon > 0$  we have

$$|\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| \leq \varepsilon \wedge |\mathcal{L}_S(\mathcal{G}, q'') - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q'')| \leq \varepsilon \Rightarrow |\mathcal{L}_S(\mathcal{G}, q') - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q')| \leq \varepsilon .$$

The above implies that for any game  $\mathcal{G}$  and for any  $\varepsilon > 0$ , we have that

$$(\forall q \in \partial \mathcal{Q}_{\mathcal{G}}) |\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| \leq \varepsilon \Rightarrow (\forall q \in \mathcal{Q}_{\mathcal{G}}) |\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| \leq \varepsilon ,$$

where  $\partial \mathcal{Q}_{\mathcal{G}}$  is the boundary of the set  $\mathcal{Q}_{\mathcal{G}}$ , i.e.,  $\partial \mathcal{Q}_{\mathcal{G}} = \{\frac{|\mathcal{NE}(\mathcal{G})|}{2^n}, 1 - \frac{1}{2^{n+1}}\}$ . From the above, the union bound, the Hoeffding's inequality and Lemma 1, we have that

$$\begin{aligned} \mathbb{P}_S[(\forall \mathcal{G} \in \mathcal{H}, q \in \mathcal{Q}_{\mathcal{G}}) |\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| \leq \varepsilon/2] \\ &= 1 - \mathbb{P}_S[(\exists \mathcal{G} \in \mathcal{H}, q \in \mathcal{Q}_{\mathcal{G}}) |\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| > \varepsilon/2] \\ &\geq 1 - \mathbb{P}_S[(\exists \mathcal{G} \in \mathcal{H}, q \in \partial \mathcal{Q}_{\mathcal{G}}) |\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| > \varepsilon/2] \\ &\geq 1 - 2d(\mathcal{H}) \mathbb{P}_S[|\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_{\mathcal{D}}(\mathcal{G}, q)| > \varepsilon/2] \\ &\geq 1 - 4d(\mathcal{H}) e^{-m\varepsilon^2/2} \\ &\geq 1 - \delta , \end{aligned}$$

where  $d(\mathcal{H}) \leq 2^{4(k+1)n(\log^2 n + \log n) + 2}$  is the number of PSNE sets that can be produced by games in  $\mathcal{H}$ , as defined in Lemma 1. The factor 2 in  $2d(\mathcal{H})$  in the union bound comes from the fact that the set  $\partial \mathcal{Q}_{\mathcal{G}}$  has exactly two elements. By solving for  $m$  in the last inequality, we get  $m \geq \frac{2}{\varepsilon^2}((4(k+1)n(\log^2 n + \log n) + 2) \log 2 + \log \frac{1}{\delta})$  as desired.

We proved so far that with probability at least  $1 - \delta$ , we have  $|\mathcal{L}_S(\mathcal{G}, q) - \mathcal{L}_D(\mathcal{G}, q)| \leq \varepsilon/2$  simultaneously for all  $\mathcal{G} \in \mathcal{H}$  and  $q \in \mathcal{Q}_G$ . Additionally, since  $(\hat{\mathcal{G}}, \hat{q})$  is the pair with minimum  $\mathcal{L}_S(\mathcal{G}, q)$  from all  $\mathcal{G} \in \mathcal{H}$  and  $q \in \mathcal{Q}_G$ , we have that

$$\begin{aligned} \mathbb{E}_D[\mathcal{L}_{\hat{\mathcal{G}}, \hat{q}}(\mathbf{x}) - \mathcal{L}_{\bar{\mathcal{G}}, \bar{q}}(\mathbf{x})] &= \mathcal{L}_D(\hat{\mathcal{G}}, \hat{q}) - \mathcal{L}_D(\bar{\mathcal{G}}, \bar{q}) \\ &\leq \mathcal{L}_S(\hat{\mathcal{G}}, \hat{q}) + \varepsilon/2 - \mathcal{L}_S(\bar{\mathcal{G}}, \bar{q}) + \varepsilon/2 \\ &\leq \varepsilon, \end{aligned}$$

with probability at least  $1 - \delta$ , which proves our claim.  $\square$

**Sufficient Samples for PSNE Recovery.** Finally, we show that if the number of samples is greater than  $\mathcal{O}(kn \log^2 n)$ , then MLE correctly recovers the PSNE with high probability.

**Theorem 3** (Sufficient samples for PSNE recovery). *Fix  $\delta, \varepsilon \in (0, 1)$ . Let  $\mathcal{H}$  be the class of LIGs with  $n$  nodes and  $k \in \{2, \dots, \lfloor \frac{9}{20}\sqrt{n} \rfloor - 1\}$  parents per node. Assume that the data distribution  $\mathcal{D} = \mathcal{P}_{\mathcal{G}^*, q^*}$  for some true game  $\mathcal{G}^* \in \mathcal{H}$  and mixture parameter  $q^* \in \mathcal{Q}_{\mathcal{G}^*}$ . Assume that  $S$  is a dataset of  $m$  joint actions (of the  $n$  players), each independently drawn from  $\mathcal{D}$ . If  $m \geq \frac{2}{\varepsilon^2}((4(k+1)n(\log^2 n + \log n) + 2)\log 2 + \log \frac{1}{\delta})$  then*

$$\mathbb{P}_S[\mathcal{NE}(\mathcal{G}^*) \subseteq \mathcal{NE}(\hat{\mathcal{G}})] \geq 1 - \delta.$$

provided that  $|\mathcal{NE}(\mathcal{G}^*)| \geq 2$  and  $\varepsilon < \beta(n, |\mathcal{NE}(\mathcal{G}^*)|, q^*)$  where<sup>4</sup>

$$\beta(n, r, q) = \frac{1}{(2n+1)\log 2} \left( q \log \frac{q}{r} + (1-q) \log \frac{1-q}{2n-r} - \frac{r-1}{r} q \log \frac{q}{r-1} - \left(\frac{q}{r} + 1 - q\right) \log \frac{1-q}{2n-r+1} \right).$$

*Proof.* Here, we follow a *worst case* approach in which we analyze the identifiability of the PSNE set of  $\mathcal{G}^*$  with respect to a game  $\mathcal{G}^-$  that has one PSNE less than  $\mathcal{G}^*$ . For our argument, showing the existence of such LIG  $\mathcal{G}^-$  is not necessary. In fact, a more general argument could be made with respect to a game that has  $k^- \geq 1$  less PSNEs than  $\mathcal{G}^*$ . The analysis for  $k^- = 1$  provides the sufficient conditions for the general case  $k^- \geq 1$ .

For clarity, let  $c(n) \equiv (2n+1)\log 2$ ,  $\bar{\mathcal{NE}} \equiv \mathcal{NE}(\hat{\mathcal{G}})$  and  $\mathcal{NE}^* \equiv \mathcal{NE}(\mathcal{G}^*)$ . Define the game  $\mathcal{G}^-$  by its PSNE set  $\mathcal{NE}^- \equiv \mathcal{NE}(\mathcal{G}^-)$  as follows. Define the set  $\mathcal{NE}^- = \mathcal{NE}^* - \{\mathbf{x}\}$  for some  $\mathbf{x} \in \mathcal{NE}^*$ . It can be easily verified that

$$\begin{aligned} |\mathcal{NE}^-| &= |\mathcal{NE}^*| - 1, & |\mathcal{NE}^* \cap \mathcal{NE}^-| &= |\mathcal{NE}^*| - 1, & |\mathcal{NE}^* \cup \mathcal{NE}^-| &= |\mathcal{NE}^*|, \\ |\mathcal{NE}^* - \mathcal{NE}^-| &= 1, & |\mathcal{NE}^- - \mathcal{NE}^*| &= 0. \end{aligned}$$

For any pair of games  $\mathcal{G}, \mathcal{G}' \in \mathcal{H}$ , let  $\mathcal{NE} \equiv \mathcal{NE}(\mathcal{G})$  and  $\mathcal{NE}' \equiv \mathcal{NE}(\mathcal{G}')$ . For any pair of games  $\mathcal{G}, \mathcal{G}' \in \mathcal{H}$ , and mixture parameters  $q \in \mathcal{Q}_G$  and  $q' \in \mathcal{Q}_{G'}$ , we have

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_{\mathcal{G}, q}}[\log p_{\mathcal{G}', q'}(\mathbf{x})] &= \sum_{\mathbf{x} \in \{-1, +1\}^n} p_{\mathcal{G}, q}(\mathbf{x}) \log p_{\mathcal{G}', q'}(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in \mathcal{NE} \cap \mathcal{NE}'} p_{\mathcal{G}, q}(\mathbf{x}) \log p_{\mathcal{G}', q'}(\mathbf{x}) + \sum_{\mathbf{x} \in \mathcal{NE} - \mathcal{NE}'} p_{\mathcal{G}, q}(\mathbf{x}) \log p_{\mathcal{G}', q'}(\mathbf{x}) \\ &\quad + \sum_{\mathbf{x} \in \mathcal{NE}' - \mathcal{NE}} p_{\mathcal{G}, q}(\mathbf{x}) \log p_{\mathcal{G}', q'}(\mathbf{x}) + \sum_{\mathbf{x} \notin \mathcal{NE} \cup \mathcal{NE}'} p_{\mathcal{G}, q}(\mathbf{x}) \log p_{\mathcal{G}', q'}(\mathbf{x}) \\ &= \frac{|\mathcal{NE} \cap \mathcal{NE}'|}{|\mathcal{NE}|} q \log \frac{q'}{|\mathcal{NE}'|} + \frac{|\mathcal{NE} - \mathcal{NE}'|}{|\mathcal{NE}|} q \log \frac{1-q'}{2^n - |\mathcal{NE}'|} \\ &\quad + \frac{|\mathcal{NE}' - \mathcal{NE}|}{2^n - |\mathcal{NE}|} (1-q) \log \frac{q'}{|\mathcal{NE}'|} + \frac{2^n - |\mathcal{NE} \cup \mathcal{NE}'|}{2^n - |\mathcal{NE}|} (1-q) \log \frac{1-q'}{2^n - |\mathcal{NE}'|}. \end{aligned} \quad (4)$$

<sup>4</sup>For all  $n$ , note that  $\beta(n, r, q) \in [(\log(1 + \frac{q}{r(1-q)}) - \frac{r-1}{r} q \log(1 + \frac{1}{(r-1)(1-q)})) / (2 \log \frac{q}{r} + \log 2), \frac{q}{2r}]$ . The minimum value is reached when  $n = \log \frac{q}{r} / \log 2$  and the maximum value is reached when  $n \rightarrow \infty$ .

Note that the pair  $(\mathcal{G}^-, q^*)$  is well defined. More formally, since  $|\mathcal{NE}^-| = |\mathcal{NE}^*| - 1$  then  $\mathcal{Q}_{\mathcal{G}^-} = ((|\mathcal{NE}^*| - 1)/2^n, 1 - 1/2^{n+1})$ . Thus,  $q^* \in \mathcal{Q}_{\mathcal{G}^*} \Rightarrow q^* \in \mathcal{Q}_{\mathcal{G}^-}$ . From eq.(4), we have

$$\begin{aligned} \mathbb{KL}(\mathcal{P}_{\mathcal{G}^*, q^*} \| \mathcal{P}_{\mathcal{G}^-, q^*}) &= \mathbb{E}_{\mathcal{P}_{\mathcal{G}^*, q^*}} [\log p_{\mathcal{G}^*, q^*}(\mathbf{x}) - \log p_{\mathcal{G}^-, q^*}(\mathbf{x})] \\ &= q^* \log \frac{q^*}{|\mathcal{NE}^*|} + (1 - q^*) \log \frac{1 - q^*}{2^n - |\mathcal{NE}^*|} \\ &\quad - \frac{|\mathcal{NE}^*| - 1}{|\mathcal{NE}^*|} q^* \log \frac{q^*}{|\mathcal{NE}^*| - 1} - \left( \frac{q^*}{|\mathcal{NE}^*|} + 1 - q^* \right) \log \frac{1 - q^*}{2^n - |\mathcal{NE}^*| + 1}. \end{aligned}$$

By the assumption in the theorem and the above, we have that

$$\begin{aligned} c(n) \varepsilon &< c(n) \beta(n, |\mathcal{NE}^*|, q^*) \\ &= \mathbb{KL}(\mathcal{P}_{\mathcal{G}^*, q^*} \| \mathcal{P}_{\mathcal{G}^-, q^*}). \end{aligned} \tag{5}$$

Note that since  $\mathcal{D} = \mathcal{P}_{\mathcal{G}^*, q^*}$  then  $\mathcal{NE}(\widehat{\mathcal{G}}) = \mathcal{NE}(\mathcal{G}^*)$  and  $\widehat{q} = q^*$ . By Lemma 2 and eq.(3), if  $m \geq \frac{2}{\varepsilon^2}((4(k+1)n(\log^2 n + \log n) + 2) \log 2 + \log \frac{1}{\delta})$  then

$$\begin{aligned} c(n) \varepsilon &\geq c(n) \mathbb{E}_{\mathcal{P}_{\mathcal{G}^*, q^*}} [\mathcal{L}_{\widehat{\mathcal{G}}, \widehat{q}}(\mathbf{x}) - \mathcal{L}_{\mathcal{G}^*, q^*}(\mathbf{x})] \\ &= \mathbb{E}_{\mathcal{P}_{\mathcal{G}^*, q^*}} [\log p_{\mathcal{G}^*, q^*}(\mathbf{x}) - \log p_{\widehat{\mathcal{G}}, \widehat{q}}(\mathbf{x})] \\ &= \mathbb{KL}(\mathcal{P}_{\mathcal{G}^*, q^*} \| \mathcal{P}_{\widehat{\mathcal{G}}, \widehat{q}}). \end{aligned}$$

Note that from the above and eq.(5), we have that  $\mathbb{KL}(\mathcal{P}_{\mathcal{G}^*, q^*} \| \mathcal{P}_{\widehat{\mathcal{G}}, \widehat{q}}) < \mathbb{KL}(\mathcal{P}_{\mathcal{G}^*, q^*} \| \mathcal{P}_{\mathcal{G}^-, q^*})$ . That is, the empirical MLE minimizer  $(\widehat{\mathcal{G}}, \widehat{q})$  is better than the pair  $(\mathcal{G}^-, q^*)$ . Therefore,  $\widehat{\mathcal{NE}}$  includes all the PSNE in  $\mathcal{NE}^*$ , i.e.,  $\mathcal{NE}^* \subseteq \widehat{\mathcal{NE}}$  and we prove our claim.  $\square$

**Remark.** A similar argument as in Theorem 3 can be used to additionally show that  $\mathcal{NE}(\widehat{\mathcal{G}}) \subseteq \mathcal{NE}(\mathcal{G}^*)$ , although the sufficient number of samples increases to  $\mathcal{O}(kn^3 \log^2 n)$  in that case. (The function  $\beta$  in such a case does not contain the  $\mathcal{O}(1/n)$  factor.)

## 5 Necessary Samples for PSNE Recovery

In this section, we show that if the number of samples is less than  $\Omega(kn \log^2 n)$ , then any conceivable method fails to recover the PSNE with probability at least 1/2.

**Theorem 4** (Necessary samples for PSNE recovery). *Let  $\mathcal{H}$  be the class of LIGs with  $n$  nodes and  $k \in \{1, \dots, n-1\}$  parents per node. Assume that the true game  $\mathcal{G}^*$  is chosen uniformly at random (by nature) from a finite subset of  $\mathcal{H}$ . Assume that the true mixture parameter  $q^*$  is known to the learner. After choosing the true game  $\mathcal{G}^*$ , nature generates a dataset  $S$  of  $m$  joint actions (of the  $n$  players), each independently drawn from  $\mathcal{P}_{\mathcal{G}^*, q^*}$ . Assume that a learner uses the dataset  $S$  in order to choose a game  $\widehat{\mathcal{G}}$ . If  $m \leq \frac{3}{2 \log 3 \log 4} (kn \log^2 n - kn \log k \log n - 2n \log 2 \log n)$  then*

$$\mathbb{P}_{\mathcal{G}^*, S}[\mathcal{NE}(\widehat{\mathcal{G}}) \neq \mathcal{NE}(\mathcal{G}^*)] \geq 1/2,$$

for any conceivable learning mechanism for choosing  $\widehat{\mathcal{G}}$ .

*Proof.* Let  $\Pi = \{\pi \mid \pi \in \{1, \dots, n\} \wedge |\pi| = k\}$ . Let  $\pi \in \Pi$  be the set of  $k$  “influential” players. Assume that nature picks  $\pi$  uniformly at random from the  $\binom{n}{k}$  elements in  $\Pi$ . For a fixed  $\pi$ , we will construct a true game  $\mathcal{G}^\pi$ . For clarity, we define  $\mathcal{G}^\pi \equiv \mathcal{G}^*$  and  $q \equiv q^*$ . The goal of the learner is to use the dataset  $S$  in order to choose a set  $\widehat{\pi}$  of  $k$  players, and to output a game  $\mathcal{G}^{\widehat{\pi}} \equiv \widehat{\mathcal{G}}$ .

For a fixed  $\pi$ , we construct a game  $\mathcal{G}^\pi$  with a single PSNE (i.e.,  $|\mathcal{NE}(\mathcal{G}^\pi)| = 1$ ) as follows. The  $k$  “influential” players do not have any parent. Furthermore, we force the “influential” players to have a best response  $-1$ , by setting their tolerance to 1. More formally,  $(\forall i \in \pi) \mathbf{w}_{i, -i} = \mathbf{0}, b_i = 1$ .

The remaining  $n - k$  “influenced” players have the  $k$  “influential” players as parents. Furthermore, we force the “influenced” players to have a best response +1. We attain this by setting their influence factor (from the  $k$  “influential” players) to  $-1$  and their tolerance to 0. More formally,  $(\forall i \notin \pi, j) w_{ij} = -1[j \in \pi]$  and  $(\forall i \notin \pi) b_i = 0$ .

Recall that the best response for all players  $i$  as in eq.(1) describes the PSNE of the game  $\mathcal{G}^\pi$ . Thus,  $\mathbf{x} \in \mathcal{NE}(\mathcal{G}^\pi)$  if and only if

$$\begin{aligned} (\forall i \in \pi) \quad x_i(-1) &\geq 0, \\ (\forall i \notin \pi) \quad x_i \left( -\sum_{j \in \pi} x_j \right) &\geq 0. \end{aligned}$$

Under the above setting, we can describe the PSNE set of the game  $\mathcal{G}^\pi$  as follows

$$\begin{aligned} (\forall i) \quad x_i^\pi &= -1[i \in \pi] + 1[i \notin \pi], \\ \mathcal{NE}(\mathcal{G}^\pi) &= \{\mathbf{x}^\pi\}. \end{aligned}$$

Since we assume a known fixed mixture parameter  $q$  and since  $|\mathcal{NE}(\mathcal{G}^\pi)| = 1$ , the PMF defined in eq.(2) reduces to

$$\begin{aligned} p_\pi(\mathbf{x}) &\equiv p_{\mathcal{G}^\pi, q}(\mathbf{x}) \\ &= 1[\mathbf{x} = \mathbf{x}^\pi] q + 1[\mathbf{x} \neq \mathbf{x}^\pi] \frac{1-q}{2^n-1}. \end{aligned}$$

Let  $\mathcal{P}_\pi$  denote the probability distribution defined by the PMF  $p_\pi(\cdot)$ . Clearly,  $\pi \neq \pi'$  if and only if  $\mathbf{x}^\pi \neq \mathbf{x}^{\pi'}$ . Thus, for all  $\pi \neq \pi'$  the Kullback-Leibler divergence is bounded as follows

$$\begin{aligned} \mathbb{KL}(\mathcal{P}_\pi \| \mathcal{P}_{\pi'}) &= \sum_{\mathbf{x} \in \{-1, +1\}^n} p_\pi(\mathbf{x}) \log p_\pi(\mathbf{x}) - \sum_{\mathbf{x} \in \{-1, +1\}^n} p_\pi(\mathbf{x}) \log p_{\pi'}(\mathbf{x}) \\ &= p_\pi(\mathbf{x}^\pi) \log p_\pi(\mathbf{x}^\pi) + \sum_{\mathbf{x} \neq \mathbf{x}^\pi} p_\pi(\mathbf{x}) \log p_\pi(\mathbf{x}) - p_\pi(\mathbf{x}^\pi) \log p_{\pi'}(\mathbf{x}^\pi) - p_\pi(\mathbf{x}^{\pi'}) \log p_{\pi'}(\mathbf{x}^{\pi'}) \\ &\quad - \sum_{\mathbf{x} \notin \{\mathbf{x}^\pi, \mathbf{x}^{\pi'}\}} p_\pi(\mathbf{x}) \log p_{\pi'}(\mathbf{x}) \\ &= q \log q + (2^n - 1) \frac{1-q}{2^n-1} \log \left( \frac{1-q}{2^n-1} \right) - q \log \left( \frac{1-q}{2^n-1} \right) - \frac{1-q}{2^n-1} \log q - (2^n - 2) \frac{1-q}{2^n-1} \log \left( \frac{1-q}{2^n-1} \right) \\ &= \frac{2^n q - 1}{2^n - 1} \left( \log q - \log \left( \frac{1-q}{2^n-1} \right) \right). \end{aligned}$$

Assume that the value of the mixture parameter (known to the learner) is  $q \equiv 2/2^n \in \mathcal{Q}_{\mathcal{G}^\pi}$ . Thus, for  $n \geq 2$  and all  $\pi \neq \pi'$  we have

$$\begin{aligned} \mathbb{KL}(\mathcal{P}_\pi \| \mathcal{P}_{\pi'}) &= \frac{\log(2^n-1) - \log(2^{n-1}-1)}{2^n-1} \\ &\leq \frac{\log 3 \log 4}{3n \log n}. \end{aligned}$$

The above bound is tight, i.e., the above becomes an equality for  $n = 2$ . Conditioned on  $\pi$ ,  $S$  is a dataset of  $m$  i.i.d. joint actions drawn from  $\mathcal{P}_\pi$ . That is,  $S \mid \pi \sim \mathcal{P}_\pi^m$ . The mutual information can be bounded by a pairwise KL-based bound [9] as follows

$$\begin{aligned} \mathbb{I}(\pi, S) &\leq \frac{1}{|\Pi|^2} \sum_{\pi \in \Pi} \sum_{\pi' \in \Pi} \mathbb{KL}(\mathcal{P}_\pi^m \| \mathcal{P}_{\pi'}^m) \\ &\leq \max_{\pi \neq \pi'} \mathbb{KL}(\mathcal{P}_\pi^m \| \mathcal{P}_{\pi'}^m) \\ &= m \max_{\pi \neq \pi'} \mathbb{KL}(\mathcal{P}_\pi \| \mathcal{P}_{\pi'}) \\ &\leq m \frac{\log 3 \log 4}{3n \log n}. \end{aligned}$$



Note that  $\hat{\pi} = \pi$  if and only if  $\mathcal{NE}(\mathcal{G}^{\hat{\pi}}) = \mathcal{NE}(\mathcal{G}^{\pi})$ . Recall that  $|\Pi| = \binom{n}{k} \geq (\frac{n}{k})^k$ . By the Fano's inequality [2] on the Markov chain  $\pi \rightarrow S \rightarrow \hat{\pi}$  we have

$$\begin{aligned} \mathbb{P}_{\mathcal{G}^*, S}[\mathcal{NE}(\hat{\mathcal{G}}) \neq \mathcal{NE}(\mathcal{G}^*)] &= \mathbb{P}_{\pi, S}[\mathcal{NE}(\mathcal{G}^{\hat{\pi}}) \neq \mathcal{NE}(\mathcal{G}^{\pi})] \\ &= \mathbb{P}_{\pi, S}[\hat{\pi} \neq \pi] \\ &\geq 1 - \frac{\mathbb{I}(\pi, S) + \log 2}{\log |\Pi|} \\ &\geq 1 - \frac{m \frac{\log 3 \log 4}{3n \log n} + \log 2}{k(\log n - \log k)} \\ &= 1/2. \end{aligned}$$

By solving the last equality, we prove our claim.  $\square$

## References

- [1] E. Brenner and D. Sontag. SparsityBoost: A new scoring function for learning Bayesian network structure. *Uncertainty in Artificial Intelligence*, pages 112–121, 2013.
- [2] T. Cover and J. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2nd edition, 2006.
- [3] D. Fudenberg and J. Tirole. *Game Theory*. The MIT Press, 1991.
- [4] J. Honorio and L. Ortiz. Learning the structure and parameters of large-population graphical games from behavioral data. *Journal of Machine Learning Research*, 16(Jun):1157–1210, 2015.
- [5] M. Irfan and L. Ortiz. On influence, stable behavior, and the most influential individuals in networks: A game-theoretic approach. *Artificial Intelligence*, 215:79–119, 2014.
- [6] M. Kearns, M. Littman, and S. Singh. Graphical models for game theory. *Uncertainty in Artificial Intelligence*, pages 253–260, 2001.
- [7] T. Neylon. *Sparse Solutions for Linear Prediction Problems*. PhD thesis, New York University, May 2006.
- [8] E. Sontag. VC dimension of neural networks. In *Neural Networks and Machine Learning*, pages 69–95. Springer, 1998.
- [9] B. Yu. Assouad, Fano, and Le Cam. In Torgersen E. Pollard D. and Yang G., editors, *Festschrift for Lucien Le Cam: Research Papers in Probability and Statistics*, pages 423–435. Springer New York, 1997.